

EXISTENCE OF COHERENT SYSTEMS.

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1. INTRODUCTION

Let C be a projective, algebraic non-singular curve of given genus g . A coherent system on C is a pair (E, V) where E is a vector bundle and V is a subspace of the space of sections of E . A coherent subsystem is a pair (E', V') where E' is a subbundle of E and $V' \subset V \cap H^0(E')$.

Given a real number $\alpha > 0$ a coherent system is said to be α -(semi)-stable if for every proper coherent subsystem

$$\frac{\deg E' + \alpha \dim V'}{\operatorname{rk} E'} < (\leq) \frac{\deg E + \alpha \dim V}{\operatorname{rk} E}.$$

Fix values of the degree d , the dimension k of V and the rank r of E as well as a rational number $\alpha > 0$. The set of coherent systems that are α -stable can be given the structure of a moduli space (see [KN] [LP]). This moduli space, if non-empty, has dimension at each point at least equal to the Brill-Noether number (see [BGMN])

$$\rho = r^2(g - 1) + 1 - k(k - d + r(g - 1)).$$

In fact, it is expected that under mild additional conditions, the moduli space will be non-empty of dimension ρ provided that ρ is positive.

Coherent systems have received a great deal of attention in the last few years. For an (almost up to date) detailed exposition of the current knowledge, the reader is advised to read the introduction in [BGMMN].

For genus one, these spaces have been described precisely in [LN]. For higher genus, the most pressing question is to have conditions that ensure the non-emptiness of these moduli spaces. Such conditions have been given for $k \leq r$ in [BG, BGMN, BGMMN]. A great deal is known for $k = r + 1$ (see [BGMN, B]). In this paper we shall concentrate on the case $k > r$ for generic curve

1.1. Theorem *Let C be a generic non-singular curve of genus $g \geq 2$. Let d, r, k be positive integers with $k > r$. Write*

$$d = rd_1 + d_2, \quad k = rk_1 + k_2, \quad d_2 < r, \quad k_2 < r$$

and all d_i, k_i non-negative integers. Assume that

$$(*)g - (k_1 + 1)(g - d_1 + k_1 - 1) \geq 1, \quad 0 \neq d_2 \geq k_2$$

$$(**)g - k_1(g - d_1 + k_1 - 1) > 1, \quad d_2 = k_2 = 0$$

$$(***)g - (k_1 + 1)(g - d_1 + k_1) \geq 1, \quad d_2 < k_2.$$

Then, for any positive value of α the moduli space of coherent systems of rank r degree d and with k sections on C is non-empty and has one component of the expected dimension ρ .

1.2. Remark *The bounds in the Theorem are those in [T2] and are not meant to be the best possible. The methods in this paper can be used to improve them although it is likely that this will require a case by case analysis (see for instance [T1] for a similar problem in the case of rank two).*

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2. BACKGROUND ON REDUCIBLE CURVES

This moduli space, if non-empty, has dimension at each point at least equal to the Brill-Noether number (see [BGMN])

$$\rho = r^2(g - 1) + 1 - k(k - d + r(g - 1)).$$

We shall use a special type of reducible curve that we shall call a chain of elliptic curves and is defined as follows:

2.1. Definition *Let $C_1 \dots C_g$ be elliptic curves. Let P_i, Q_i be generic points in C_i . Then C_0 is the chain obtained by gluing the elliptic curves by identifying the point Q_i in C_i to the point P_{i+1} in C_{i+1} , $i = 1 \dots g - 1$. When there is no danger of confusion, we shall write P, Q instead of P_i, Q_i .*

We shall be using the following well-known fact

2.2. Lemma *Let C be an elliptic curve and L a line bundle of degree d on C . One can define a subspace of dimension k of sections of L by specifying the k distinct (minimum) desired vanishings of a basis of the subspace at two different points P, Q so that the sum of the corresponding vanishings at P and Q is $d - 1$. In the case when $L = \mathcal{O}(aP + (d - a)Q)$ two of the vanishings could be chosen to be $a, d - a$ adding to d rather than $d - 1$. These are the only two vanishings that can add up to d if P, Q are generic.*

Similarly, let E be a vector bundle obtained as the sum of r line bundles of degree d . Then one can find a space V of dimension k of sections of E with desired vanishings a_1, \dots, a_k and b_1, \dots, b_k respectively at two points P, Q if $a_i + b_{k-i} \leq d - 1$ and each possible value of a_i, b_i appears at most r times.

If E is an indecomposable vector bundle of rank r and degree $rd_1 + d_2$ with $0 < d_2 < r$, there are no sections whose orders of vanishing at two points add up to a

number greater than d_1 . The space of sections vanishing to order a at P and $d_1 - a$ at Q has dimension d_2 .

The proof is left to the reader (or see [T1])

When dealing with reducible curves, the notion of a line bundle and a space of its sections needs to be replaced by the analogous concept of limit linear series as introduced by Eisenbud and Harris. A similar definition can be given for vector bundles (cf[T2]). For the convenience of the reader, we reproduce this definition here

2.3. Limit linear series *A limit linear series of rank r , degree d and dimension k on a chain of M (not necessarily elliptic) curves consists of data I,II below for which data III, IV exist satisfying conditions a)-c)*

I) *For every component C_i , a vector bundle E_i of rank r and degree D_i and a k -dimensional space V_i of sections of E_i*

II) *For every node obtained by gluing Q_i and P_{i+1} , an isomorphism of the projectivisation of the fibers $(E_i)_{Q_i}$ and $(E_{i+1})_{P_{i+1}}$*

III) *A positive integer b*

IV) *For every node obtained by gluing Q_i and P_{i+1} bases $s_{Q_i}^t, s_{P_{i+1}}^t$, $t = 1 \dots k$ of the vector spaces V_i, V_{i+1} of I .*

Subject to the conditions

a) $\sum_{i=1}^M D_i - r(M-1)b = d$

b) *The orders of vanishing at Q_i, P_{i+1} of the sections of the chosen bases satisfy $\text{ord}_{Q_i} s_{Q_i}^t + \text{ord}_{P_{i+1}} s_{P_{i+1}}^t \geq b$*

c) *Sections of the vector bundles $E_i(-bP_i), E_i(-bQ_i)$ are completely determined by their values at the nodes.*

3. SKETCH OF THE PROOF

We now give a general outline of the proof of the theorem.

We want to show that for any positive α and given degree d , rank r and number of sections k , the set of α -stable coherent systems satisfying the conditions of the Theorem is non-empty. We shall choose E to be a stable vector bundle (with the usual meaning for stability) and V a generic subspace of dimension k of its sections. Because of the stability of E , for every subbundle E' of E one has the inequality

$$\frac{\deg E'}{\text{rk} E'} < \frac{\deg E}{\text{rk} E}.$$

Therefore, in order to prove the stability of the coherent system, it suffices to show that for every subbundle E' of E , the dimension k' of the vector space $H^0(E') \cap V$ satisfies

$$\frac{k'}{\text{rk} E'} \leq \frac{k}{\text{rk} E}.$$

In fact, if E is stable then (E, V) is α -stable for arbitrarily large α if and only if the condition above is satisfied.

We need to prove the result for the generic curve. It suffices to prove it for a particular curve and a coherent system which is a member of a family of dimension precisely ρ on that curve. This follows from the fact that the dimension of a family of coherent systems on a family of curves $\mathcal{C} \rightarrow B$ is at least $\dim B + \rho$. If the dimension of one of the fibers over B is precisely ρ , then the family projects onto the base (see for instance [S, T2, T1] for similar arguments).

The curve we shall use will be a singular curve of the special type that we described 2.1. We then construct a limit coherent system on the chain of elliptic curves and show as outlined above that no subbundle has enough sections to contradict stability.

4. THE LIMIT LINEAR SERIES

As in the statement of the Theorem, write $d = rd_1 + d_2, k = rk_1 + k_2$. Let h be the greatest common divisor of d and r and $d = \bar{d}h, r = \bar{r}h$. We describe next limit linear series on C_0 , we need to distinguish between various cases depending on the relative values of d_2, k_2 . The number b that appear in III of 2.3, will be taken to be d_1 in all cases.

Assume first $d_2 < k_2$ and $g - d_1 + k_1 \geq 1$. The latter condition is equivalent to the fact that a generic vector bundle of rank r and degree d has fewer than k sections.

We start by describing the vector bundles and gluings that appear in the definition of limit linear series. We leave the description of the spaces of sections to later.

On C_1 take the vector bundle to be the direct sum of h generic vector bundles of rank \bar{r} and degree \bar{d} .

On the curve $C_i, i = 2, \dots, k_1 + 2$ take the vector bundle

$$\mathcal{O}((2i - 3)P + (d_1 - 2i + 3)Q)^{k_2 - d_2} \oplus L_1^i \oplus \dots \oplus L_{r - k_2 + d_2}^i$$

where the L_j^i are generic line bundles of degree d_1 . Consider the space \bar{W}_1 of dimension d_2 of sections of E_1 with maximum vanishing d_1 at Q_1 . Take the gluing of E_{1, Q_1} with E_{2, P_2} so that the image of W_{1, Q_1} by the map $H^0(E_1(-d_1 Q_1) \otimes \mathcal{O}_C \rightarrow E_{1, Q_1}(-d_1 Q_1))$ lies inside the space of dimension $r - k_2 + d_2$ of the fiber of E_2 at P_2 generated by the subbundle $L_1^2 \oplus \dots \oplus L_{r - k_2 + d_2}^2$.

Take the gluing at the next k_1 nodes so that the subbundle $L_1^i \oplus \dots \oplus L_{r - k_2 + d_2}^i$ on the curve C_i glues with the analogous subbundle on the curve C_{i+1} .

Consider inside the space of sections of $E_2(-(d_1 - 1)Q_2)$ the subspace \bar{W}_2 of those gluing at P_2 with \bar{W}_1 . Let \tilde{W}_2 be the subspace of sections of $E_2(-(d_1 - 1)Q_2 - P_2)$. Define $W_2 = \bar{W}_2 + \tilde{W}_2$.

For $i \geq 3$, define W_i by induction in the following way: Consider the space of sections of $E_i(-(i - 2)P_i - (d_1 - i + 1)Q_i)$. It has dimension r and maps bijectively

on the fibers of P_i and Q_i . Take as W_i the subspace of dimension k_2 that glues with W_{i-1} at P_i .

Note that by our assumption (***) $(k_1 + 1)(g + k_1 - d_1) + 1 \leq g$. On the curves

$$C_{(k_1+1)\alpha+\beta+1}, \quad \alpha = 1, \dots, g + k_1 - d_1 - 1, \quad \beta = 1, \dots, k_1 + 1$$

take the following vector bundles:

If $\beta = 1$ take

$$\mathcal{O}((k_1\alpha + 1)P + (d_1 - k_1\alpha - 1)Q)^{k_2} \oplus L_1^i \oplus \dots \oplus L_{r-k_2}^i$$

For $\beta = 2, \dots, k_1 + 1$ take

$$\mathcal{O}((k_1\alpha + 2\beta - 1)P + (d_1 - k_1\alpha - 2\beta + 1)Q)^r.$$

For $\beta = 2, \dots, k_1 + 1$, the gluing is generic.

For $\beta = 1, \alpha = 1$ take the gluing so that the subbundle

$$\mathcal{O}((k_1 + 1)P + (d_1 - k_1 - 1)Q)^{k_2}$$

glues with W_{k_1+2} .

Define W_i as before where for $i = (k_1 + 1)\alpha + 1 + \beta$ the spaces of sections to be considered are

$$H^0(E_i(-(\alpha k_1 + \beta - 1)P - (d_1 - \alpha k_1 - \beta)Q), \beta \neq 1$$

$$H^0(E_i(-(\alpha k_1 + 1)P - (d_1 - \alpha k_1 - 1)Q), \beta = 1.$$

For $\beta = 1, \alpha > 1$ take the gluing so that the fiber at $Q_{\alpha(k_1+1)+1}$ of the subbundle

$$\mathcal{O}((k_1\alpha + 1)P + (d_1 - k_1\alpha - 1)Q)^{k_2}$$

glues with $W_{\alpha(k_1+1)+2}$.

On the remaining curves, the vector bundles are direct sums of generic line bundles of degree d_1 and the gluing is generic.

One can still define the subspace W_i as before. Write $t = g + k_1 - d_1$ and use as space of sections $H^0(E_i(-(i - t - 1)P - (d_1 - i + t)Q))$.

The resulting vector bundle is stable because the restriction of the vector bundle to each component is semistable and the destabilizing subbundles in the various components do not glue with each other (see [T3, T4])

We want to compute the dimension of the set of such vector bundles. We need to add the dimensions of the families of the restrictions of the vector bundles to each component and the dimensions of the families of gluings. We then subtract the dimensions of the families of automorphisms of the vector bundles restricted to each component and finally add one as the resulting bundle being stable, it has a one dimensional family of automorphisms.

The first bundle moves in an h dimensional family and has an h dimensional family of automorphisms. The next $k_1 + 1$ bundles move in a $[r - k_2 + d_2]$ -dimensional family and have a $[(k_2 - d_2)^2 + r - k_2 + d_2]$ -dimensional family of automorphisms. The first

gluing varies in a $[d_2(r - (k_2 - d_2)) + r(r - d_2)]$ -dimensional family. The next k_1 gluings vary in a $[(r - k_2 + d_2)^2 + r(k_2 - d_2)]$ -dimensional family.

On the curves corresponding to $\beta = 1$, the vector bundle varies in a $[r - k_2]$ -dimensional family and has a $[(k_2)^2 + r - k_2]$ -dimensional family of automorphisms while the gluing depends on $k_2^2 + r(r - k_2)$ parameters.

On the curves corresponding to $\beta > 1$, the vector bundles are completely determined, they have an r^2 -dimensional family of endomorphisms and the gluings are free.

On the remaining curves, the vector bundles depend on r parameters, have an r -dimensional family of endomorphisms and the gluings are free.

Therefore, the dimension of the family is

$$\begin{aligned} & [h - h] + [r - k_2 + d_2 - (k_2 - d_2)^2 - (r - k_2 + d_2) + d_2(r - (k_2 - d_2)) + r(r - d_2)] \\ & + k_1[r - k_2 + d_2 - (k_2 - d_2)^2 - (r - k_2 + d_2) + (r - (k_2 - d_2))^2 + r(k_2 - d_2)] \\ & (g + k_1 - d_1 - 1)[r - k_2 - k_2^2 - (r - k_2) + k_2^2 + r(r - k_2)] + (g + k_1 - d_1 - 1)k_1(-r^2 + r^2) \\ & + (g - (k_1 + 1)(g + k_1 - d_1) - 1)(r - r + r^2) + 1 = \rho \end{aligned}$$

We need to define the spaces of sections on every component of the curve and check that this gives rise to only one limit series for each bundle as defined above. As pointed out before, we take $b = d_1$ in III of 2.3.

Similarly to the way we defined W_i , we shall define spaces of sections W'_i this time by descending induction for $i = k_1 + 2, \dots, 1$. Define first

$$W'_{k_1+2} = H^0(E_{k_1+2}(-(2k_1 + 1)P - (d_1 - 2k_1 - 1)Q)).$$

By definition,

$$E_{k_1+2} = (\mathcal{O}((2k_1 + 1)P + (d_1 - 2k_1 - 1)Q))^{k_2 - d_2} \oplus L_1^{k_1+2} \oplus \dots \oplus L_{r-k_2+d_2}^{k_1+2}$$

where the last $r - k_2 + d_2$ terms are generic line bundles of degree d_1 . Therefore, $E_{k_1+2}(-(2k_1 + 1)P - (d_1 - 2k_1 - 1)Q)$ consists of a direct sum of $k_2 - d_2$ copies of the trivial line bundle and $r - k_2 + d_2$ generic line bundles of degree zero. It follows that W'_{k_1+2} has dimension $k_2 - d_2$.

For $i < k_1 + 2$, define W'_i by descending induction as the space of sections of $E_i(-(i + k_1 - 1)P - (d_1 - i - k_1)Q)$ that glues with W'_{i+1} at Q_i .

For $i = 1$, the space of sections of the limit linear series is

$$H^0(E_1(-(d_1 - k_1)Q_1)) \oplus W'_1.$$

The vanishings of the sections of this space at P_1 are

$$0, 1, k_1 - 1, k_1$$

where each vanishing is repeated r times except for the last one which is repeated k_2 times. The vanishings of the sections of this space at Q_1 are

$$d_1, d_1 - 1, \dots, d_1 - k_1, d_1 - k_1 - 1$$

where each vanishing is repeated r times except for the first one which is repeated d_2 times and the last one which is repeated $k_2 - d_2$ times.

For $i = 2, \dots, k_1 + 2$, the space of sections is

$$W_i \oplus H^0(E_i(-(i-1)P - (d_1 - 2i + 3)Q)) \oplus H^0(L_1^i \oplus \dots \oplus L_{r+d_2-k_2}^i(-(2i-3)P - (d_1 - 2i + 2)Q)) \\ \oplus H^0(E_i(-(2i-2)P - (d_1 - k_1 - i + 1)Q)) \oplus W'_i.$$

For $i = 2$, the vanishings of the sections of this space at P_2 are $0, 1, 2, \dots, k_1, k_1 + 1$ where each number is repeated r times except for the first one which appears d_2 times and the last one that appears $k_2 - d_2$ times.

The vanishings of the sections of this space at P_i for $i > 2$ are

$$i - 2, i - 1, \dots, 2i - 4, 2i - 3 \dots k_1 + i - 2, k_1 + i - 1$$

where each vanishing is repeated r times except for the first one which is repeated k_2 times, the vanishing $2i - 4$ which is repeated $r + d_2 - k_2$ times and the last one which is repeated $k_2 - d_2$ times.

The vanishings of the sections of this space at Q_i are

$$d_1 - i + 1, d_1 - i \dots d_1 - 2i + 4, d_1 - 2i + 3, d_1 - 2i + 2 \dots, d_1 - k_1 - i + 1, d_1 - k_1 - i$$

where each vanishing is repeated r times except for the first one which is repeated k_2 times, the vanishing $d_1 - 2i + 2$ which is repeated $r + d_2 - k_2$ times and the last one which is repeated $k_2 - d_2$ times.

For $i = \alpha(k_1 + 1) + \beta + 1$ the space of sections is

$$W_i \oplus H^0(E_i(-(k_1\alpha + \beta)P - (d_1 - k_1\alpha - 2\beta + 1)Q)) \oplus \\ \oplus H^0(E_i(-(k_1\alpha + 2\beta)P - (d_1 - k_1(\alpha + 1) - \beta - 1)Q)).$$

The vanishings of the sections of this space at P_i are

$$i - \alpha - 2, i - \alpha - 1, \dots, k_1\alpha + 2\beta - 3, k_1\alpha + 2\beta - 1, k_1\alpha + 2\beta, \dots, k_1(\alpha + 1) + \beta$$

where each vanishing is repeated r times except for the first one which is repeated k_2 times. The vanishings of the sections of this space at Q_i are

$$d_1 - i + \alpha + 1, d_1 - i + \alpha, \dots, d_1 - k_1\alpha - 2\beta + 1, d_1 - k_1\alpha - 2\beta - 1, \dots, d_1 - k_1(\alpha + 1) - \beta - 1$$

where each vanishing is repeated r times except for the first one which is repeated k_2 times.

Write $t = g + k_1 - d_1$. then for $i > t(k_1 + 1) + 1$, the space of sections is

$$W_i \oplus H^0(E_i(-(i-t)P - (d_1 - i + t - k_1)Q))$$

The vanishings of the sections of this space at P_i are

$$i - t - 1, i - t, \dots, i - t + k_1 - 1$$

where each vanishing is repeated r times except for the first one which is repeated k_2 times. The vanishings of the sections of this space at Q_i are

$$d_1 - i + t, d_1 - i + t - 1, \dots, d_1 - i + t - k_1$$

where each vanishing is repeated r times except for the first one which is repeated k_2 times.

One checks then that the vanishings at P_1 and Q_g are the smallest possible, namely

$$0, 1, \dots, k_1 - 1, k_1$$

each with multiplicity r except for k_1 which has multiplicity k_2 . Moreover, the sum of the vanishings at Q_i and P_{i+1} of corresponding sections is the minimum possible namely $b = d_1$. Therefore, deforming one of the E_i at one of the components C_i to a more general bundle would decrease the vanishing of the limit linear series at Q_i and would therefore make it impossible to prolong the limit linear series till C_g . Similarly, deforming one of the gluings between $E_{i-1, Q_{i-1}}$ and E_{i, P_i} to a more general gluing would also decrease the vanishing at Q_i . Therefore, this family is not part of a family of limit linear series of dimension larger than ρ . It follows that it can be deformed to the generic curve and will be part of a family of dimension ρ on the generic curve.

It remains to show that no coherent subsystem (E', V') contradicts stability for any value of α or equivalently that $\frac{k'}{r'} \leq \frac{k}{r}$. Assume by contradiction that this subsystem existed. Then the restriction to the curve C_1 would give a coherent subsystem of the restriction of (E, V) . Note that V is a generic subspace of sections of $E(-(d_1 - k_1 - 1)Q)$. The latter is a generic vector bundle of degree $r(k_1 + 1) + d_2 > k$. Then, Thm. 5.4 of [LN] applies and such a V' cannot exist.

Consider now the case in which $0 \neq d_2 \geq k_2$ and assume that $g + k_1 - d_1 \geq 2$. The latter condition is equivalent to the fact that the generic vector bundle of rank r and degree d has fewer than k sections.

On C_1 take the vector bundle to be the direct sum of h generic vector bundles of rank \bar{r} and degree \bar{d} .

On the curve C_2 take the vector bundle

$$\mathcal{O}(d_1 Q_2)^{k_2} \oplus L_1^2 \oplus \dots \oplus L_{r-k_2}^2$$

where the L_i^2 are generic line bundles of degree d_1 . Take the gluing at P_2 so that the subbundle $\mathcal{O}(d_1 Q_2)^{k_2}$ glues inside the fiber at Q_1 of the space of dimension d_2 of sections of E_1 with maximum vanishing d_1 at Q_1 . The fiber of $L_1^2 \oplus \dots \oplus L_{r-k_2}^2$ at P_2 intersects $H^0(E_1(-d_1 Q_1))$ in a vector space of dimension $d_2 - k_2$. Consider the space of sections of $H^0(E_2(-(d_1 - 1)Q_2))$ that glue with this space. They generate a space of dimension $d_2 - k_2$ at Q_2 . Denote this by \bar{V} . This is a space of dimension k_2 .

Define $W_2 = H^0(E_2(-d_1 Q_2))$.

On the curve C_i , $i = 3, \dots, k_1 + 2$ take the vector bundle

$$\mathcal{O}((2i - 4)P + (d_1 - 2i + 4)Q)^{r+k_2-d_2} \oplus L_1^i \oplus \dots \oplus L_{d_2-k_2}^i$$

where the L_j^i are generic line bundles of degree d_1 . At P_3 take the gluing so that $L_1^3 \oplus \dots \oplus L_{d_2-k_2}^3$ on the curve C_3 glues with \bar{V} defined above.

For $i \geq 3$, take the gluing so that the subbundle $L_1^i \oplus \dots \oplus L_{d_2-k_2}^i$ on the curve C_i glues with the corresponding subbundle on the curve C_{i+1} .

Define W_i by induction as the space of sections of $H^0(E_i(-(i-3)P - (d_1-i+2)Q))$ that glues with W_{i-1} at P_i .

On the curves

$$C_{(k_1+1)\alpha+\beta+1}, \quad \alpha = 1, \dots, g + k_1 - d_1 - 2, \quad \beta = 1, \dots, k_1 + 1$$

take the following vector bundles:

If $\beta = 1$ take

$$\mathcal{O}((k_1\alpha + 1)P + (d_1 - k_1\alpha - 1)Q)^{k_2} \oplus L_1^i \oplus \dots \oplus L_{r-k_2}^i$$

For $\beta = 2, \dots, k_1 + 1$ take

$$\mathcal{O}((k_1\alpha + 2\beta - 2)P + (d_1 - k_1\alpha - 2\beta + 2)Q)^r.$$

For $\beta = 2, \dots, k_1 + 1$, the gluing is generic. For $\beta = 1, \alpha = 1$ take the gluing so that $\mathcal{O}((k_1\alpha)P + (d_1 - k_1\alpha)Q)^{k_2}$ glues with W_{k_1+2} .

Define W_i inductively using the space of sections

$$H^0(E_i(-[k_1\alpha P + (d_1 - k_1\alpha)Q])), \quad \beta = 1$$

$$H^0(E_i(-[(k_1\alpha + \beta - 2)P + (d_1 - k_1\alpha - 2)Q])), \quad \beta > 1$$

that glues with the previous W_{i-1} .

For $\beta = 1, \alpha > 1$, glue the fiber at P_i , $i = (k_1 + 1)\alpha + 2$ of $\mathcal{O}((k_1\alpha + 1)P + (d_1 - k_1\alpha - 1)Q)^{k_2}$ with the fiber at Q_{i-1} of W_{i-1} .

On the remaining curves, the vector bundles are direct sums of generic line bundles of degree d_1 and the gluings are generic.

Define $t = g + k_1 - d_1 - 1$. One can again define W_i inductively using the space of sections of

$$H^0(E_i(-[(i - t - 2)P + (d_1 - i + t + 1)Q])).$$

that glue with the previous W_{i-1} .

The resulting vector bundle is stable because the restriction of the vector bundle to each component is semistable and the destabilizing subbundles in the various components do not glue with each other.

In order to compute the dimension of the set of such vector bundles, we need to add the dimensions of the families of the vector bundles on each component, the dimensions of the families of gluings, subtract the dimensions of the families of automorphisms of the vector bundle restricted to each component and finally add one as the resulting bundle being stable, it has a one dimensional family of automorphisms.

The first bundle moves in an h dimensional family and has an h dimensional family of automorphisms.

The second one moves in an $[r - k_2]$ -dimensional family and has a $[(k_2)^2 + r - k_2]$ -dimensional family of automorphisms.

Each of the next k_1 bundles moves in a $[d_2 - k_2]$ -dimensional family and has a $[(r + k_2 - d_2)^2 + d_2 - k_2]$ -dimensional family of automorphisms. The first gluing varies in a $[d_2 k_2 + r(r - k_2)]$ -dimensional family. The next k_1 gluings vary in $[(d_2 - k_2)^2 + r(r - d_2 + k_2)]$ -dimensional families.

On the curves corresponding to $\beta = 1$, the vector bundle varies in a $[r - k_2]$ -dimensional family and has a $[(k_2)^2 + r - k_2]$ -dimensional family of automorphisms while the gluing depends on $[k_2^2 + r(r - k_2)]$ parameters.

On the curves corresponding to $\beta > 1$, the vector bundles are completely determined, they have an r^2 -dimensional family of endomorphisms and the gluings are free.

On the remaining curves, the vector bundles depend on r -parameters, have an r -dimensional family of endomorphisms and the gluings are free.

Therefore, the dimension of the family is

$$\begin{aligned} & [h - h] + [r - k_2 - (k_2)^2 - (r - k_2) + d_2 k_2 + r(r - k_2)] \\ & + k_1 [d_2 - k_2 - (r - (d_2 - k_2))^2 - (d_2 - k_2) + (d_2 - k_2)^2 + r(r - d_2 + k_2)] \\ & (g + k_1 - d_1 - 2)[r - k_2 - k_2^2 - (r - k_2) + k_2^2 + r(r - k_2)] + (g + k_1 - d_1 - 2)k_1(-r^2 + r^2) \\ & + (g - (k_1 + 1)(g + k_1 - d_1 - 1) - 1)(r - r + r^2) + 1 = \rho \end{aligned}$$

We need to define the spaces of sections on every component of the curve and check that this gives rise to only one limit series for each bundle as defined above.

Define

$$W'_{k_1+2} = H^0(E_{k_1+2}(-2k_1P - (d_1 - 2k_1)Q)).$$

By definition,

$$E_{k_1+2} = (\mathcal{O}((2k_1)P + (d_1 - 2k_1)Q))^{r+k_2-d_2} \oplus L_1^{k_1+2} \oplus \dots \oplus L_{d_2-k_2}^{k_1+2}$$

where the last $d_2 - k_2$ terms are generic line bundles of degree d_1 . Therefore, $E_{k_1+2}(-(2k_1)P - (d_1 - 2k_1)Q)$ consists of a direct sum of $r + k_2 - d_2$ copies of the trivial line bundle and $d_2 - k_2$ generic line bundles of degree zero. It follows that W'_{k_1+2} has dimension $r + k_2 - d_2$.

For $i < k_1 + 2$, define W'_i by descending induction as the space of sections of $E_i(-(i + k_1 - 2)P - (d_1 - i - k_1 + 1)Q)$ that glues with W'_{i+1} at Q_i .

Define now the spaces of sections of the limit linear series.

For $i = 1$, the space of sections is

$$H^0(E_1(-(d_1 - k_1 + 1)P)) \oplus W'_1.$$

The vanishings of the sections of this space at P_1 are

$$0, 1, k_1 - 1, k_1$$

where each vanishing is repeated r times except for the last one which is repeated k_2 times. The vanishing of the sections of this space at Q_1 are

$$d_1, d_1 - 1, \dots, d_1 - k_1 + 1, d_1 - k_1$$

where each vanishing is repeated r times except for the first one which is repeated d_2 times and the last one which is repeated $r + k_2 - d_2$ times.

For $i = 2$, the space of sections is

$$W_2 \oplus \bar{V} \oplus H^0(E_2(-P_2 - (d_1 - k_1)Q_2)) \oplus W'_2$$

The vanishing of this space of sections at P_2 is $0, 1, \dots, k_1 - 1, k_1$, where each vanishing is repeated r times except for the first one that appears d_2 times and the last one that appears $r - d_2 + k_2$ times. At Q_2 , the vanishing is $d_1, d_1 - 1, \dots, d_1 - k_1 - 1$ where each vanishing appears r times except for the first one which appears k_2 times, the second one which appears $d_2 - k_2$ times and the last one that appears $r + k_2 - d_2$ times.

For $i = 3, \dots, k_1 + 2$, the space of sections is

$$W_i \oplus H^0(E_i(-(i-2)P - (d_1 - 2i + 4)Q)) \oplus H^0(E_i(-(2i-3)P - (d_1 - k_1 - i + 2)Q)) \\ \oplus H^0(L_1^i \oplus \dots \oplus L_{d_2 - k_2}^i(-(2i-4)P - (d_1 - 2i + 3)Q)) \oplus W'_i.$$

The vanishings of the sections of this space at P_i are

$$i - 3, i - 2, \dots, 2i - 6, 2i - 5, 2i - 4, \dots, k_1 + i - 3, k_1 + i - 2$$

where each vanishing is repeated r times except for the first one which is repeated k_2 times, the vanishing $2i - 5$ which is repeated $d_2 - k_2$ times and the last one which is repeated $r + k_2 - d_2$ times. The vanishings of the sections of this space at Q_i are

$$d_1 - i + 2, d_1 - i + 1, \dots, d_1 - 2i + 5, d_1 - 2i + 4, d_1 - 2i + 3, \dots, d_1 - k_1 - i + 2, d_1 - k_1 - i + 1$$

where each vanishing is repeated r times except for the first one which is repeated k_2 times, the vanishing $d_1 - 2i + 3$ which is repeated $d_2 - k_2$ times and the last one which is repeated $r + k_2 - d_2$ times.

For $i = \alpha(k_1 + 1) + \beta$ the space of sections is

$$W_i \oplus H^0(E_i(-(k_1\alpha + \beta - 1)P - (d_1 - (k_1\alpha + 2\beta - 2)Q))) \oplus \\ \oplus H^0(E_i(-(k_1\alpha + 2\beta - 1)P - (d_1 - (k_1(\alpha + 1) + \beta)Q))).$$

The vanishings of the sections of this space at P_i are

$$i - \alpha - 3, \dots, k_1\alpha + 2\beta - 4, k_1\alpha + 2\beta - 2, k_1\alpha + 2\beta - 1, \dots, k_1(\alpha + 1) + \beta - 1$$

where each vanishing is repeated r times except for the first one which is repeated k_2 times. The vanishings of the sections of this space at Q_i are

$$d_1 - i + \alpha + 2, \dots, d_1 - k_1\alpha - 2\beta + 3, d_1 - k_1\alpha - 2\beta + 2, d_1 - k_1\alpha - 2\beta, \dots, d_1 - k_1(\alpha + 1) - \beta$$

where each vanishing is repeated r times except for the first one which is repeated k_2 times.

Write $t = g + k_1 - d_1 - 1$. then for $i > t(k_1 + 1) + 1$, the space of sections is

$$W_i \oplus H^0(E_i(-(i - t - 1)P - (d_1 - i + t - k_1 + 1)Q))$$

The vanishings of the sections of this space at P_i are

$$i - t - 2, i - t - 1, \dots, i - t + k_1 - 2$$

where each vanishing is repeated r times except for the first one which is repeated k_2 times. The vanishings of the sections of this space at Q_i are

$$d_1 - i + t + 1, d_1 - i + t, \dots, d_1 - i + t - k_1 + 1$$

where each vanishing is repeated r times except for the first one which is repeated k_2 times.

As in the case $d_2 < k_2$, this defines a limit linear series with $b = d_1$ which is not part of a larger family of limit linear series, therefore it deforms to the generic curve.

It only remains to prove the claim that for every coherent subsystem (E', V') of rank r' , degree d' and with $\dim V' = k'$, $k'/r' \leq k/r$. Suppose by contradiction that there exists a subsystem with $k'/r' > k/r$. Note that by our choice of gluing, the space of sections that we take on the first vector bundle E_1 is in fact a generic subspace of dimension k of $H^0(E_1(-(d_1 - k_1)Q_1))$. Then if $k_2 < d_2$ or $k_2 = d_2, h = 1$ from [LN] Th 5.4, the restriction of (E', V') to C_1 cannot exist.

It remains to deal with the case $k_2 = d_2, h \neq 1$. Then $E_1 = F_1 \oplus \dots \oplus F_h$ where each F_j is an indecomposable bundle of rank \bar{r} and degree $\bar{d} = \bar{r}d_1 + \bar{d}_2$, $k_2 = d_2 = h\bar{d}_2$. As E_1 is semistable, the slope of E' is at most the slope of E_1 . Again, V' is a subspace of the space of sections of $E'(-(d_1 - k_1)Q_1)$. The latter is a vector bundle of rank r' and degree $d' - r'(d_1 - k_1) = r'k_1 + d'_2$. Then, $k' \leq r'k_1 + d'_2$. Hence,

$$\frac{k'}{r'} \leq \frac{r'k_1 + d'_2}{r'} \leq \frac{rk_1 + d_2}{r} = \frac{rk_1 + k_2}{r} = \frac{k}{r}.$$

When $d_2 = k_2 = 0$, take on C_1 the direct sum of r generic line bundles of degree d_1 . On

$$C_{\alpha k_1 + \beta + 1}, \quad \alpha = 0, \dots, g - 2 - d_1 + k_1, \quad \beta = 1, \dots, k_1$$

take

$$\mathcal{O}((2\beta + \alpha(k_1 - 1) - 1)P + (d_1 - 2\beta - \alpha(k_1 - 1) + 1)Q)^r$$

On the remaining curves take direct sums of generic vector bundles of degree d_1 .

As spaces of sections of the limit linear series take on $C_{\alpha k_1 + \beta + 1}$

$$\begin{aligned} & H^0(E_i(-(i - 2 - \alpha)P - (d_1 - 2\beta - \alpha(k_1 - 1) + 1)Q)) \oplus \\ & \oplus H^0(E_i(-(2\beta + \alpha(k_1 - 1))P - (d_1 - i - k_1 + \alpha + 1)Q)) \end{aligned}$$

Write $t = g - d_1 + k_1 - 1$. For $i \geq k_1 t + 2$, take as space of sections $H^0(E_i(-(i - t - 1)P - (d_1 - i + t - k_1 + 1)Q))$. We leave it to the reader to check that one gets

a limit linear series with the value of the parameter b in the definition being d_1 and the dimension of the family being ρ .

Consider now the case in which $d + r(1 - g) \geq k$. Then, the vector bundle is going to be generic while the limit linear series for a fixed bundle will move in a space of dimension equal to the dimension of the grassmannian $Gr(k, d + r(1 - g))$. The details are as follow:

On the curve C_1 take as before a generic sum of h vector bundles of rank \bar{r} and degree \bar{d} . On the curves C_2, \dots, C_g take the vector bundle to be a direct sum of r generic line bundles of degree d_1 .

The integer b that appears in III of the definition of limit linear series above will be taken again to be d_1 . Glue the vector bundles E_i to E_{i+1} at the point Q_i, P_{i+1} by using generic gluing.

The vector bundle obtained in this way is stable because all the restrictions to the elliptic components are semistable and the destabilising line bundles for each component do not glue with each other (see [T3, T4]). Moreover it corresponds to a generic point of a component of the moduli space of vector bundles of this rank and degree. Hence, it moves in a family of dimension $r^2(g - 1) + 1$.

Take a space of sections of E_1 of dimension k generic inside $E_1(-(g - 1)Q_1)$. For each of the vanishings $g - 1, g, \dots, g + k_1 - 2$, this space contains r independent sections with these orders of vanishing. It contains a k_2 -dimensional space of sections with vanishing $g + k_1 - 1$.

On E_i one must then take r -dimensional spaces of sections with vanishing $g - i + j$ at Q_i and $d_1 - g + i - 1 - j$ at P_i , $j = 0 \dots k_1 - 1$ as well as a k_2 dimensional space of sections with vanishing $d_1 - g + i - k_1 - 1$ at P_i and $g - i + k_1$ at Q_i that glues at P_i with the corresponding space at Q_{i-1} .

One can check that this gives rise to a limit linear series. The sum of the vanishings at Q_{i-1} and P_i of corresponding sections adds up to the minimum required (d_1) while the vanishing at Q_g is $0, 1 \dots k_1 - 1, k_1$ where each number is repeated r times except for the last one which is repeated d_2 times. These are the minimum possible vanishings for the sections of a k dimensional linear series at a point. Hence, the series is completely determined by the choice of the subspace of sections on the first curve. This first choice is equivalent to the choice of a k -dimensional vector space of $H^0(E_1(-(g - 1)Q_1))$. The latter is a vector space of dimension $d - r(g - 1)$. It follows that, for a fixed vector bundle, there is a $[k(d - r(g - 1) - k)]$ -dimensional family of limit linear series. Then, the set of coherent systems moves in a family of dimension $r^2(g - 1) + 1 + k(d - r(g - 1) - k) = \rho$.

Assume that the coherent system contained a coherent subsystem corresponding to a subbundle E' of rank r' with k' sections where $\frac{k'}{r'} > \frac{k}{r}$. If $k < d + r(1 - g)$ or $(r, d) = 1$, Theor.5.4 of [LN] shows that the restriction to C_1 cannot exist. When $k = d + r(1 - g)$ and $(r, d) \neq 1$, the result follows with arguments similar to the previous cases.

This concludes the proof of the Theorem.

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